

RENORMALIZATION WITHOUT REGULARIZATION AND R-OPERATION

Fyodor A. Lunev

Department of Physics, Moscow State University, Moscow, 119899, Russia

Definition of Feynman integrals as solutions of some well defined systems of differential equations is proposed. This definition is equivalent to usual one but needs no regularization and application of *R*-operation. It is argued that proposed renormalization scheme maintains all symmetries that can be maintained in perturbative quantum field theory, and also is convenient for practical calculations.

1. Motivation.

Though fundamental results in renormalization theory were obtained many years ago in classical works of Feynman, Tomonaga, Schwinger, Dyson, Salam, Bogolubov, Parasiuk, Hepp, and Zimmermann¹, renormalization problems continue to attract the attention of theorists. In particular, during last twenty years very many papers were devoted to investigations of various renormalization schemes.

Of course, all known renormalization schemes are equivalent, in principal, at perturbative level. However, their practical value is different. To be useful, the renormalization scheme must

- maintain symmetries
- be convenient for practical calculations

Usually, it is hard to satisfy both this conditions. Indeed, for application of standard renormalization schemes one must

- fix some regularization
- apply *R*-operation with some fixed parameters

But

- symmetries can be broken at both stages
- regularization usually hampers the evaluation of diagrams

So, I believe, it is interesting to formulate renormalization scheme that needs no any regularization and application of *R*-operation at all.

Such renormalization scheme is presented in this report. It is equivalent to usual *R*-operation scheme. But "equivalent" doesn't mean "the same". Indeed, in standard *R*-operation renormalization scheme one must, first, regularize initial divergent (in general) Feynman integral. Then it is necessary to use rather complicated subtraction prescription (forest formula) to obtain finite result. Nothing similar is needed in my renormalization scheme. To obtain finite expression for given Feynman integral, one must only solve some well defined differential equations. Neither any regularization, nor any manipulations with counterterm diagrams are needed to obtain finite result.

¹Beautiful account of foundations and modern achievements of renormalization theory can be found, for instance, in monographs [1]

2. Differential equations for definition and evaluation of Feynman integrals.

2.A. The simplest example.

To clarify the main idea of presented work, let us consider the simplest divergent (Euclidean) Feynman diagram (see Fig.1). For simplicity, we will consider massless case. Then in coordinate space this diagram is well defined function



Figure 1: The simplest divergent Feynman diagram

$$\tilde{\Gamma}(x) = \frac{(2\pi)^4}{(x^2)^2} \quad (1)$$

But the function $\tilde{\Gamma}(x)$ is not a distribution. So its Fourier image $\Gamma(p)$ doesn't exist. But $\tilde{\Gamma}(x)$ is defined as distribution on the set

$$S_0 = \{\phi(x) \in S : \phi(0) = 0\} \quad (2)$$

where S is Schwartz space. So the problem of renormalization theory can be formulated as follows:

to define $\tilde{\Gamma}^{ren}(x) \in S'$ in such a way that

$$\left. \begin{array}{l} \tilde{\Gamma}^{ren} \\ S_0 \end{array} \right| = \tilde{\Gamma} \quad (3)$$

One notes that the distribution $\tilde{\Gamma}^{ren}(x)$, defined as a solution of equation

$$x^2 \tilde{\Gamma}^{ren}(x) = \frac{(2\pi)^4}{x^2}, \quad (4)$$

satisfies the conditions formulated above. (This follows from Hormander's theorem about possibility of dividing by polynomial in S'). So we see that it is very natural to define $\Gamma^{ren}(p)$ as a solution of the differential equation

$$\Delta \Gamma^{ren}(p) = -\frac{4\pi^2}{p^2} \quad (5)$$

that is equivalent to (4). The general solution of equation (5) from S' is given by formula

$$\Gamma^{ren} = -\pi^2 \ln \frac{p^2}{\mu^2} \quad (6)$$

where μ^2 is the constant of integration. This expression coincide with usual one for "bubble" diagram in Fig.1. We see that it is possible to obtain the standard result for diagram under consideration without using of any regularization and any subtraction procedure.

2.B. General formalism

Without loss of generality, we will consider only 1PI (Euclidean) diagram Γ_L without internal vertexes. In coordinate space

$$\tilde{\Gamma}_L(x_1, \dots, x_n; \{m_{ij}^2\}) = \prod_{\text{all lines of } \Gamma} D(x_i - x_j; m_{ij}^2) \quad (7)$$

where L is the number of loops and m_{ij}^2 are masses in propagators.

Below we will interpret m_{ij}^2 as the square of some Euclidean two dimensional vector. Then we can write

$$m_{ij}^2 = (m_{ij,1})^2 + (m_{ij,2})^2 \quad (8)$$

Further, let us define

$$\hat{D}(x, u) = \int d^2 m e^{i\vec{u}\vec{m}} D(x, m) = \int d^4 p d^2 m \frac{e^{ipx + i\vec{u}\vec{m}}}{p^2 + m^2} = \frac{16\pi^3}{(x^2 + m^2)^2} \quad (9)$$

and

$$\begin{aligned} \hat{\Gamma}_n(x_1, \dots, x_n; \{u_{ij}^2\}) &\equiv \int \prod_{\text{all lines of } \Gamma} d^2 m_{ij} \exp \left(i \sum_{\text{all lines of } \Gamma} \vec{m}_{ij} \vec{u}_{ij} \right) \tilde{\Gamma}_n(x_1, \dots, x_n; \{m_{ij}^2\}) \\ &= (16\pi^3)^N \frac{1}{P(x_1, \dots, x_n; \{u_{ij}^2\})} \end{aligned} \quad (10)$$

where N is total number of lines in diagram Γ and P is the polynomial:

$$P = \prod_{\text{all lines of } \Gamma} [(x_i - x_j)^2 + u_{ij}^2] \quad (11)$$

One can see that $\hat{\Gamma}_L \notin S'$, but $\hat{\Gamma}_L$ is defined as distribution on the space of test functions

$$S_0 = \left\{ \phi \in S' : \begin{array}{l} \phi = 0, \quad \partial_\alpha \phi = 0, \dots \\ \Big|_{x_i = x_j} \end{array} \right\} \quad (12)$$

The function $\hat{\Gamma}_L^{ren}$ can be defined as the prolongation of $\hat{\Gamma}_L$ on the whole space S .

Now we can give the following inductive definition of $\hat{\Gamma}_L^{ren}$.

Let $\hat{\Gamma}_{L-1}^{ren}$ is already defined. Then $\hat{\Gamma}_L^{ren}$ is defined by the following equations in coordinate space:

$$[(x_i - x_j)^2 + u_{ij}^2]^2 \hat{\Gamma}_L^{ren}(x_1, \dots, x_n, \{u_{kl}^2\}) = (16\pi^3) \hat{\Gamma}_{L-1}^{ren}(x_1, \dots, x_n, \{u_{kl}^2\}) \Big|_{(kl) \neq (ij)} \quad (13)$$

or, if

$$\Delta_{(p_i - p_j)m_{ij}} = \sum_{\alpha=1}^4 \left(\frac{\partial}{\partial p_{i,\alpha}} - \frac{\partial}{\partial p_{j,\alpha}} \right)^2 + \sum_{\alpha=1}^2 \frac{\partial^2}{\partial m_{ij,\alpha}^2} \quad (14)$$

by the following equivalent equations in the momentum space:

$$[\Delta_{(p_i - p_j)m_{ij}}]^2 \Gamma_L^{ren}(p_1, \dots, p_n, \{m_{kl}^2\}) = 16\pi^3 \delta(\vec{m}_{ij}) \Gamma_{L-1}^{ren}(p_1, \dots, p_n, \{m_{kl}^2\}) \Big|_{(kl) \neq (ij)} \quad (15)$$

with boundary conditions

$$\lim_{|p| \rightarrow \infty} \frac{1}{p^{\omega(\Gamma)+\epsilon}} \Gamma_L^{ren} = 0; \quad \lim_{m_{ij} \rightarrow \infty} \frac{1}{m_{ij}^{\omega(\Gamma)+\epsilon}} \Gamma_L^{ren} = 0. \quad (16)$$

One can prove that Γ_L^{ren} is defined by equations (15) with boundary conditions (16) up to polynomial of the degree $\omega(\Gamma_L)$ with respect to p_1, \dots, p_n , just as in usual formulation of the renormalization theory.

In general, the following theorem of equivalence is valid:

Let Γ_L^Λ is the diagram regularized by cutoff at $|p_i| = \Lambda$, and Γ_L^{ren} is obtained from Γ_L^Λ by means of usual R-operation. Then Γ_L^{ren} satisfies equations (15).

The proof of this theorem is given in [2].

2.C. Symmetries.

In quantum field theory any symmetry corresponds to certain Ward identity, and to maintain the symmetry in given renormalization scheme means to fix the freedom in definition of divergent Feynman integrals in such a way that the corresponding Ward identities are satisfied.

But in our renormalization scheme arbitrariness in definition of divergent Feynman diagram is not fixed *a priori*. So, if there exists the renormalization scheme with counterterms that are polynomial with respect to masses and compatible with Ward identities, that, due to equivalence theorem,² formulated above, the equations (15) are also compatible with Ward identities. So we can state that *proposed renormalization scheme maintains all symmetries that can be conserved in perturbative quantum field theory*.

2.D. Application to evaluation of concrete diagrams.

For the lack of the place, we cannot give here example of non-trivial application of proposed renormalization scheme to evaluation of concrete Feynman diagrams. So we give only some notes concerning possible advantages of our approach in comparison with dimensional regularization scheme that now is the most popular one in practical calculations.

Feynman integrals are rather complicated multiple ones. When one evaluates such integrals in non-integer dimension, then, after the first integration one obtains usually rather complicated non elementary function that hampers further integration. In our approach, one always works in four dimensions with mathematically well defined objects that need no any regularization. This allows sometimes to carry out exact evaluation of Feynman diagrams in four dimensions in situations when exact evaluation in noninteger dimensions is impossible. The example of such calculations in the case of two loop self-energy diagram with three propagators is given in ([3]).

3. Second order equations for definition of Feynman integrals.

In our approach divergent as well as convergent Feynman diagrams are defined by equations (15) with boundary conditions (16). There exists also other possibilities for definition of Feynman integrals as the solutions of the systems of differential equations. For instance, if one considers m^2 as the square of the *four* dimensional vector (rather than two dimensional as above), then, at least in the case $\omega(\Gamma_L) < 2$, one can obtain the following equations:

$$\Delta_{(p_i-p_j)m_{ij}} \left(\frac{\Gamma_L}{m_{ij}^2} \right) = -(2\pi)^2 \delta(\vec{m}) \Gamma_L \Big|_{m_{ij}=0} \quad (17)$$

²Strictly speaking, for applicability of equivalence theorem the given renormalization scheme must be equivalent to cutoff one up to finite renormalization. To author's knowledge, all known renormalization schemes satisfies this condition.

whereas the function $\Gamma_L \Big|_{m_{ij}=0}$ satisfies the equations

$$\Delta_{p_i-p_j} \Gamma_L \Big|_{m_{ij}=0} = -\frac{1}{(2\pi)^2} \Gamma_{L-1} \quad (18)$$

(In RHS of equation(18) Γ_{L-1} means the initial diagrams Γ_L without the propagator with the mass m_{ij}).

The equation (17) and (18) are the second order ones whereas equations (15) are of the forth order. This fact simplifies the investigation of the corresponding Feynman integrals. In particular, one can easy to obtain from (17) the formula that connects Feynman diagrams with massive and massless lines:

$$\Gamma_L = \frac{m_{ij}^2}{\pi^2} \int d^4 p' \frac{1}{[(p-p')^2 + m_{ij}^2]^3} \Gamma_L(p', \dots) \Big|_{m_{ij}=0} \quad (19)$$

There exist also many other equivalent definitions of divergent Feynman integrals by means of second order systems of differential equations. All of them can be obtained by means of various modifications of methods presented in section 2.

4. Conclusion.

Now it is unclear, whether our approach to renormalization theory has principal advantages in comparison with standard formulation. But, at least, calculations, presented in [2, 3], show that our approach gives new effective methods of evaluation Feynman integrals. So author believes that proposed formalism will be useful in various investigations in quantum field theory.

Author is indebted to D.J.Broadhurst, A.I.Davydychev and V.A.Smirnov for valuable discussions and comments.

References

- [1] N.N.Bogolubov, D.V.Shirkov, Introduction to the Quantum Field Theory, Willey-Intersc., 1980; J.C.Collins, Renormalization, Cambridge Univ. Press, 1984; V.A.Smirnov, Renormalization and Asymptotic Expansions, Birkhauser, 1992.
- [2] F.A. Lunev, Phys. Rev., D50 (1994) 6589
- [3] F.A. Lunev, Preprint MSU-PHYS-HEP-Lu3/94, 1994 (to be published in Phys. Rev. D).